Mathematics 222B Lecture 9 Notes

Daniel Raban

February 15, 2022

1 Solvability for Elliptic Operators

1.1 The Dirichlet problem and energy estimates for elliptic operators

We are looking at a second order, scalar, elliptic operator P (we will sometimes use L, which Evans' textbook uses):

$$Pu = -\partial_j a^{j,k}\partial_k u) + b^j \partial_j u + cu.$$

For ellipticity, we will assume that $a = [a^{j,k}]$ is a positive definite matrix, and we will further assume that $a \succeq \lambda I$ for some $\lambda > 0$ (i.e. all eigenvalues of a are $\geq \lambda$). For the purposes of this lecture, we will assume that $a, b, c \in L^\infty(U)$, where U is a bounded domain with C^1 boundary.

Last time, we looked at the Dirichlet boundary value problem

$$\begin{cases} Pu = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Recall that we may assume $g = 0$ be working with u minus some extension of g.

By the regularity assumptions on the coefficients a, b, c, $P : H^1(U) \to H^{-1}(U)$. Recall that $H^{-1}(U) = \{f = f_0 + \sum_{i=1}^d \partial_{x^i} f_i : f_0 f_i \in L^2\}$ and that $W_0^{k,p}(U)^* = W^{-k,p'}(U)$. The norm for this space is

$$\|f\|_{H^{-1}} = \inf_{f = f_0 + \sum_{i=1}^d \partial_{x^i} f_i} \left\{ \left(\|f_0\|_{L^2}^2 + \sum_{i=1}^d \|f_i\|_{L^2}^2 \right)^{1/2} \right\}.$$

To build in the Dirichlet boundary condition $u|_{\partial U} = 0$, restrict P to $P : H_0^1(U) \to H^1(U)$ (here, H_0^1 is the set of H^1 functions with 0 trace). To understand the solvability of P (i.e. existence and uniqueness), we want to understand if P is 1 to 1 and onto. We will use a priori estimates.

Last time, we proved the following a priori estimate.

Lemma 1.1 (Energy estimate)**.** *There exist $C > 0, \gamma > 0$ such that for $u \in H_0^1(U)$,*

$$\|u\|_{H^1(U)} \le C\|Pu\|_{H^{-1}(U)} + \gamma\|u\|_{L^2(U)}.$$

The proof was by integration by parts.

Recall that in order to prove existence statements with a priori estimates, we also needed to think about the dual problem for the adjoint P^*. (In finite dimensional linear algebra, $Ax = y$ has a solution x if and only if $r \in \operatorname{ran} A = {}^{\perp}(\ker A^*)$. For P as above, let's compute P^* with respect to $\langle u, v\rangle = \int uv\,dx$:

$$\int \partial_j uv\,dx = -\int u\partial_j v\,dx,$$

so

$$P^* = -\partial_j(a^{j,k}\partial_k u) - \partial_j(b^j u) + cu,$$

where we are assuming everything is real-valued. Note that the energy estimate also applies to P^*.

1.2 Case 1: Both P and P^* obey good a priori estimates

In our discussion of Sobolev spaces, we introduced the following lemma from functional analysis.

Lemma 1.2. *Let X, Y be Banach spaces, and let $P : X \to Y$ be a bounded, linear operator. If $\|u\|_X \le C\|Pu\|_Y$, then*

(i) $\ker P = \{0\}$

(ii) For every $g \in X^*$, there exists a $v \in Y^*$ such that $P^*v = g$ ($\operatorname{ran} P^* = X^*$) and $\|v\|_{X^*} \le C\|g\|_{X^*}$.

If $\|v\|_{Y^*} \le C'\|P^*v\|_{X^*}$, then

(i) $\ker P^* = \{0\}$

(ii) For every $f \in Y$, there is a $u \in X$ such that $Pu = f$ ($\operatorname{ran} P = Y$) and $\|u\|_X \le C'\|f\|_Y$.

Remark 1.1. In our previous proof, we assumed that X is reflexive to reduce (ii) to (i), but this assumption can be dropped. To see this argument, look for the "closed range theorem." The key idea is that $\overline{\operatorname{ran} P} = {}^{\perp}(\ker P^*)$.

We want to apply this lemma to our P, $X = H_0^1$, and $Y = H^{-1}(U)$. In this setting, $X^* = H^{-1}(U) = Y$, and $Y^* = H_0^1(U) = X$.

In the energy estimate, we have an extra term $\gamma\|u\|_{L^2(U)}$ in the bound. For now, we will get rid of it by cheating. We will deal with it in full later. Here is when we have the energy estimate with $\gamma = 0$:

Lemma 1.3. *If $b = 0$ and $c = 0$, i.e. $Pu = -\partial_j(a^{j,k}\partial_j u)$, then the energy estimate holds with $\gamma = 0$.*

Proof. By density, $u \in C_0^\infty$.

$$\begin{aligned}\int_U Puu\,dx &= \int_U -\partial_j(a^{j,k}\partial_k u)u\,dx \\ &= \int_U a^{j,k}\partial_j u\partial_k u\,dx \\ &\geq \lambda \int)U|Du|^2\,dx\end{aligned}$$

Using Friedrich's inequality,

$$\geq C\int_U |u|^2\,dx.$$

As in the proof of the energy estimate, we cancel a factor of $\|u\|_{H^1}$ on both sides of the inequality to get the result. ☐

Remark 1.2. Since P^* has the same form with the same constants, this condition gives the energy estimate with $\gamma = 0$ for P^*, as well.

Theorem 1.1. *For every $f \in H^{-1}(U)$, there exixsts a unique $u \in H_0^1(U)$ such that $-\partial_j(a^{j,k}\partial_j u) = f$ in U.*

Remark 1.3. For the proof of this, Evans' textbook uses the Lax-Milgram lemma, but our lemma is actually stronger.

1.3 Case 2: General P

To obtain stronger results for our general problem, we will develop tools which are specifically useful for this problem. In particular, we will discuss Fredholm theory.

Recall the notion of a compact operator $K : X \to Y$ from functional analysis: $K(\overline{B}_X)$ is compact, where $B_X = \{x \in X : \|x\| < 1\}$.

Lemma 1.4.

(o) For $K : X \to Y$, K is compact if and only if K^* is compact.

(i) (Solvability of $(I + K)x = y$): Let $K : X \to X$ be compact, and let $T = I + K$.

- *(a)* $\ker(I + K)$ *is finite dimensional.*
- *(b) There exists an $n_0 \geq 1$ such that* $\ker(I + K)^n = \ker(I + K)^{n_0}$ *for $n \geq n_0$.*
- *(c)* $\operatorname{ran}(I + K)$ *is closed, so* $\operatorname{ran}(I + K) = {}^\perp(\ker(! + K^*))$.

(d) $\dim\ker(I+K) = \dim\ker(I+K^*)$.

Remark 1.4. Part (d) is the general equivalent of the fact that in finite dimensional linear algebra, the row rank of a matrix is equal to the column rank of a matrix. This statement is that $\mathrm{index}(I+K) = 0$, where the index of an operator is the difference of these two quantities. The index tends to be very stable under perturbation.

Proof. For the proof when X is a Hilbert space, see the appendix of Evans' textbook. What is the idea? Here is how to think about compact operators: Notice that if A has $\dim\operatorname{ran} A < \infty$, then A is compact. Also notice that if $K_n \to K$ in the operator norm topology on $\mathcal{L}(X,Y)$, then K is compact. Combining these two facts tells us that the closure of the set of finite rank operators is a subset of the compact operators; in separable Hilbert spaces, this is what all compact operators look like. $\square$

Why is this lemma relevant for us? Take any general

$$Pu = -\partial_j(a^{j,k}\partial_k u) + b^j partial_j u + cu.$$

In general, the energy estimate gives

$$\|u\|_{H_0^1(U)} \leq C\|Pu\|_{H^{-1}(U)} + \gamma\|u\|_{L^2(U)}.$$

But if we consider instead $(P+\mu I)u = -\partial_j(a^{j,k}\partial_k u) + b^j$
$partial_j u + (c+\mu)u$ with $\mu \gg 1$, then we can remove γ on the right hand side.

Indeed,

$$\int (P+\mu)u\,dx = \underbrace{\int -\partial_j a^{i,k}\partial_k u\,dx}_{\geq \lambda\int |Du|^2\,dx} + b, c \text{ terms} + \int \mu u^2\,dx,$$

where the $\int \mu u^2\,dx$ term is favorable if $\mu > 0$. By case 1, for μ sufficiently positive, for all $f \in H^{-1}$, ther exists a unique $u \in H_0^1$ such that

$$(P+\mu I)u = f.$$

We then have a well-defined map $(P+\mu I)^{-1} : H^{-1}(U) \to H_0^1(U)$. Now go back to

$$(P+\mu)u - \mu u = Pu = f.$$

Apply $(P+\mu)^{-1}$ to get

$$u - \mu(P+\mu)^{-1}u = (P+\mu)^{-1}f.$$

By Rellich-Kondrachov (recalling that U is bounded), the embedding $\iota : H_0^1(U) \to L^2$ is compact. From this, it follows that

$$(P+\mu)^{-1} : L^2(U) \to H^{-1}(U) \xrightarrow{(P+\mu)^{-1}} H^1(U) \to L^2(U)$$

is compact (since $A \circ K$ or $K \circ A$ is compact whenever A is bounded and linear and K is compact). Thus, $-\mu(P+\mu)^{-1} : L^2(U) \to L^2(U)$ is compact. Thus, our repackaging of the problem,

$$u - \mu(P+\mu)^{-1}u = (P+\mu)^{-1}f,$$

is of the form $(I+K)x = y$.

Theorem 1.2 (Fredholm alternative)**.** *Let P be as before, and let U be a bounded domain with C^1 boundary.*

(i) Exactly one of the following holds:

(a) (Solvability) For all $f \in H^{-1}(U)$, there exists a unique $u \in H_0^1(U)$ such that $Pu = f$, and there exists a $C > 0$ independent of u, f such that $\|u\|_{H^1(U)} \leq C\|f\|_{H^{-1}(U)}$.

(b) (Existence of nonzero homogeneous solution) There exists a nonzero $u \in H_0^1(U)$ (or equivalently in $L^2(U)$) such that $Pu = 0$.

(ii) If (b) holds, then $\dim\ker P < \infty$ *and* $\dim\ker P^* < \infty$*. Given $f \in H^{-1}(U)$, there exists a $u \in H_0^1(U)$ such that $Pu = f$ if and only if $\langle f, v\rangle = 0$ for all $v \in \ker P^*$.*

Remark 1.5. While our initial approach didn't really care about boundedness, this approach essentially relies on this condition.

Remark 1.6. Part (ii) is a statement about norms. This will be an exercise and follows from compactness.

Remark 1.7. Here is a very nice consequence of this theorem. Take

$$\widetilde{P}u = -\partial_j(a^{j,k}\partial_k u) + b^j\partial_j u.$$

There is a weak maximum principle which says that

$$\sup_{\overline{U}} |u| = \sup_{\partial U} |u|.$$

This gives uniqueness in this Dirichlet problem. Then the Fredholm alternative gives us solvability from the uniqueness. We will properly discuss this later, when we go over maximum principles.